Throughput Maximization with Short- and Long-Term Jain's Index Guarantees in OFDMA Systems

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***Abstract*—In wireless resource allocation, improving system throughput and simultaneously enhancing user fairness are two fundamental but contradictory objectives. As for fairness, both short-term and long-term fairness are of significant importance. However, less effort has been dedicated to explore the optimal tradeoff between system throughput and the two mentioned fairness in terms of widely used Jain's index. In this context, we aim to maximize system throughput subject to constraints on both short-term and long-term fairness in single cell downlink OFDMA systems. The difficulty of this issue lies in that the considered subchannel and slot allocation problem is a non-linear integer programming problem, and furthermore seems to be non-causal. To overcome these challenges, we first relax the integer variables. Second, we prove that short-term fairness ensures long-term fairness so that the long-term fairness constraint is redundant and can be removed. Third, the problem is decomposed into a sequence of short-term convex optimization problems that can be easily solved. Numerical results show that the proposed method achieves a good suboptimal solution with small deviations from the optimal relaxed system throughput and the Jain's index constraint.**

I. Introduction

Improving system throughput as well as enhancing user fairness have attracted considerable attention in wireless resource allocation in recent years [1]. However, these two aims are not well compatible, i.e., favoring one often results in the starvation of the other [1], [2]. In particular, allocating the resources to the most favored users improves the system throughput, but discriminates the users in bad conditions. As such, the tradeoff between system throughput and user fairness has been extensively investigated [1]–[4].

In the literature, most of the emphasis has been placed on maximizing the sum of well-defined utilities to trade off throughput and fairness. For instance, log-utility is adopted to achieve proportional fairness [3]. In addition, one can utilize more general α-fair utility ($\alpha \geq 0$), under which the bias towards throughput or fairness can be easily controlled by adapting α [1], [4]. Particularly, $\alpha = 0$, $\alpha = 1$ and $\alpha \to \infty$ correspond to throughput maximization, proportional fairness and Max-Min fairness, respectively. Unfortunately, the tradeoff achieved by these utility based policies is not optimal from the perspective of Jain's index, where Jain's index is a predominantly utilized fairness measure due to its highly intuitive explanation[1] [2], [5]. In other words, either throughput or Jain's index of such tradeoff can be further improved without degrading the other. On account of this drawback, the optimal tradeoff between throughput and long-term fairness in terms of Jain's index is studied in [2] over static channels. Besides, there are also many studies focusing on maximizing throughput with long-term fairness constraints by other fairness measures [6], [7].

Nevertheless, on the fairness aspect, both long-term and short-term fairness are greatly important, which pay more attention to time average and instantaneous throughput, respectively[2] [8], [9]. More deeply, the relation between such two fairness is intuitively assumed in [8] as *short-term fairness implies long-term fairness*, which has been used in many studies since then such as [9]. However, the mathematical interpretation of this phenomenon is still lacking from the Jain's index point of view.

Inspired by this situation, the throughput maximization problem with constraints on both short-term and long-term Jain's index in downlink Orthogonal-Frequency-Division-Multi-Access (OFDMA) systems is investigated in this paper. The considered subchannel and slot allocation problem is challenging because it is a Non-linear Integer Programming (NIP) problem. Furthermore, the long-term fairness constraint couples all the instantaneous resource allocations, which means that the instantaneous resource allocation must take into account the Channel State Information (CSI) in the whole time interval. In other words, the problem is also non-causal. To overcome these two obstacles, we relax the integer variables, and remove the long-term fairness constraint by proving that short-term fairness ensures long-term fairness. Accordingly, the considered problem can be decomposed and easily solved with small deviations from the optimal system throughput of the relaxed problem and the constraint of Jain's index.

The remainder of this paper is organized as follows. Section II introduces the system model. In section III, the considered problem is formulated and settled. Finally, we present the

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[1]One can simply interpret the value of Jain's index, bounded in $[0, 1]$, as the fraction of satisfied users. As a result, a larger Jain's index corresponds to a fairer resource allocation. In peculiar, the fairest case appears when all the users receive the same throughput, which makes the Jain's index equals 1.

[2]It is worth noting that elastic services care long-term fairness, while real-time services are sensitive to short-term fairness.

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numerical results and conclude the paper in section IV and section V, respectively.

II. System Model

Consider a downlink OFDMA system with K users served by one Base Station (BS). The wireless channel is characterized as a frequency-selective channel, which is divided into M independent parallel frequency-flat subchannels. Rayleigh block fading model is assumed, where the block length is equal to the duration of N time slots. Here, a slot consists of several OFDMA symbols [6]. In particular, the channel conditions remain constant during a block of time but vary independently over different blocks and different subchannels with $\mathcal{CN}(0,1)$. Assume that the BS knows the CSI of all users over all subchannels. Our observation time contains L ($L>1$) blocks. Then, the short-term and long-term performance are measured in one block of time and L blocks of time, respectively.

The instantaneous channel fading coefficient between the BS and user k ($k=1,2,\cdots,K$) over subchannel m ($m=1,2,\cdots,M$) at the nth ($n=1,2,\cdots,LN$) slot is denoted by $h_{k,m}(n)$, which stays invariant over $(l-1)N<n\le lN$ for all $l=1,2,\cdots,L$, but varies independently over different blocks and subchannels with $\mathcal{CN}(0,1)$. For simplicity, we use $h_{k,m}^l$ and $r_{k,m}^l$ to denote the channel coefficient and instantaneous achievable rate between the BS and user k over subchannel m during the lth block, respectively. Here, $r_{k,m}^l = W\log(1+\gamma|h_{k,m}^l|^2)$, where W and γ are the bandwidth of each subchannel and the transmit signal-noise-ratio (SNR) over each subchannel, respectively. Without loss of generality, we let $W=1$ and $\gamma=1$ for normalization.

Let $Q_{k,m}^l$ be the number of slots allocated to user k over subchannel m during the lth block. Accordingly, we derive the short-term throughput of user k during the lth block, denoted by $\overline{r_k^l}$, as $\overline{r_k^l}=\frac{1}{N}\sum_{m=1}^{M} r_{k,m}^l Q_{k,m}^l$. We utilize R_k^l to stand for the average throughput of user k in the duration from the starting time to the end of the lth block ($l=1,2,\cdots,L$), i.e.,

$$R_k^l=\frac{1}{l}\sum_{i=1}^{l}\overline{r_k^i}. \tag{1}$$

In particular, we define $R_k^0=0$ for all $k=1,2,\cdots,K$. Obviously, R_k^L represents the long-term throughput of user k. Then, according to the definition of Jain's index in [5], the short-term Jain's index in the lth block, denoted by J^l, and the long-term Jain's index, denoted by J_{Long}, are given in Eq. (2) and Eq. (3), respectively.

$$J^l=\frac{\left(\sum\limits_{k=1}^{K}\overline{r_k^l}\right)^2}{K\sum\limits_{k=1}^{K}\overline{r_k^l}^2} \tag{2}$$

$$J_{Long}=\frac{\left(\sum\limits_{k=1}^{K}R_k^L\right)^2}{K\sum\limits_{k=1}^{K}\left(R_k^L\right)^2} \tag{3}$$

III. Throughput Maximization with Fairness Constraints

In this section, we formulate, relax, simplify, decompose and solve the throughput maximization problem with constraints on both short-term and long-term Jain's index in subsections III-A, III-B, III-C, III-D and III-E, respectively.

A. Problem formulation

The aim of resource allocation in this paper is maximizing the long-term throughput subject to constraints on both short-term and long-term Jain's index, each of which should be greater than or equal to a predefined threshold J_T ($0\le J_T\le 1$). Particularly, we formulate the problem as follows

$$\mathbf{P1}:\quad \max_{Q_{k,m}^l}\ \sum_{k=1}^{K}R_k^L=\frac{1}{LN}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}r_{k,m}^l Q_{k,m}^l \tag{4}$$

$$\text{s.t.}\quad Q_{k,m}^l=0,1,\cdots,N,\forall m,k,l \tag{5}$$

$$\sum_{k=1}^{K}Q_{k,m}^l\le N,\forall m,l \tag{6}$$

$$J^l\ge J_T,\forall l \tag{7}$$

$$J_{Long}\ge J_T, \tag{8}$$

where J^l and J_{Long} are given in Eq. (2) and Eq. (3), respectively.

There are $K\times M\times L$ integer variables in the subchannel and slot allocation problem **P1**, where each variable has $N+1$ possible values. Besides, the constraints in Eq. (7) and Eq. (8) are non-linear. Hence, it is difficult to find out the optimal solution. In the next subsection, we relax this problem into a continuous optimization problem.

B. Problem relaxation

We introduce a continuous variable $p_{k,m}^l$, bounded in $[0,1]$, to replace the term $\frac{Q_{k,m}^l}{N}$, where $p_{k,m}^l$ can be interpreted as the utilization fraction of subchannel m by user k during the lth block. By this relaxation, we derive $\overline{r_k^l}=\sum_{m=1}^{M}r_{k,m}^l p_{k,m}^l$ and $R_k^l=\frac{1}{l}\sum_{i=1}^{l}\overline{r_k^i}=\frac{1}{l}\sum_{i=1}^{l}\sum_{m=1}^{M}r_{k,m}^l p_{k,m}^l$. Accordingly, we present the relaxed problem of **P1** as follows

$$\mathbf{P2}:\quad \max_{p_{k,m}^l}\ \sum_{k=1}^{K}R_k^L=\frac{1}{L}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}r_{k,m}^l p_{k,m}^l \tag{9}$$

$$\text{s.t.}\quad p_{k,m}^l\ge 0,\forall m,k,l \tag{10}$$

$$\sum_{k=1}^{K}p_{k,m}^l\le 1,\forall m,l \tag{11}$$

$$(7)\text{ and }(8). \tag{12}$$

The optimal solution of **P2** is used to determine the subchannel and slot allocations as will be explained in subsection III-E.

The challenge of solving problem **P2** lies in constraint (8), where long-term fairness is related to resource allocations in all blocks. In other words, we need to take L blocks as a whole to carry out the resource allocation. However, this is impractical because future CSI is not available in advance but only instantaneous one can be obtained [10]. In the following subsection, we simplify this problem by removing redundant constraints.

C. Problem simplification

We first prove that the constraint in Eq. (8) is redundant. To derive this, we present the following theorem.

Theorem 1: If the short-term Jain's index in each block is larger than or equal to $J_T (0 \leq J_T \leq 1)$, the long-term Jain's index is also larger than or equal to J_T.

Proof: We need to show that Eq. (8) is guaranteed by Eq. (7). To obtain this, we use mathematical induction to prove

$$\frac{\left(\sum\limits_{k=1}^{K} R_k^l\right)^2}{K \sum\limits_{k=1}^{K} \left(R_k^l\right)^2} \geq J_T \tag{13}$$

for all $l = 1, 2, \cdots, L$, where the special case $l = L$ demonstrates $J_{Long} \geq J_T$.

First, Eq. (13) holds for $l = 1$ because R_k^1 equals $\overline{r_k^1}$ for all $k = 1, 2, \cdots, K$.

Second, we prove that Eq. (13) is guaranteed for l ($l = 2, 3, \cdots, L$), if Eq. (13) holds for $l - 1$.

At the beginning, we present the following expression that can be easily derived from Eq. (1)

$$R_k^l = \frac{l-1}{l} R_k^{l-1} + \frac{1}{l}\overline{r_k^l},\ l = 1, 2, \cdots, L. \tag{14}$$

According to Eq. (14), we only need to show

$$H = \frac{H_{\text{up}}}{H_{\text{down}}} = \frac{\left(\sum\limits_{k=1}^{K} \left(\theta R_k^{l-1} + (1-\theta)\overline{r_k^l}\right)\right)^2}{K\left(\sum\limits_{k=1}^{K} \left(\theta R_k^{l-1} + (1-\theta)\overline{r_k^l}\right)^2\right)} \geq J_T, \tag{15}$$

where $0 \leq \theta \leq 1$ represents the term $\frac{l-1}{l}$ in Eq. (14).

Minkowski inequality given by Eq. (16) is utilized to prove this.

$$\left(\sum_{k=1}^{K} |a_k + b_k|^t\right)^{\frac{1}{t}} \leq \left(\sum_{k=1}^{K} |a_k|^t\right)^{\frac{1}{t}} + \left(\sum_{k=1}^{K} |b_k|^t\right)^{\frac{1}{t}}, \forall\, t \geq 1 \tag{16}$$

In Eq. (16), we let $t = 2$, $a_k = \theta R_k^{l-1}$ and $b_k = (1-\theta)\overline{r_k^l}$, where both θR_k^{l-1} and $(1-\theta)\overline{r_k^l}$ are nonnegative. Then, squaring and multiplying by K both sides of the inequality yields

$$H_{\text{down}} = K\left(\sum_{k=1}^{K} \left(\theta R_k^{l-1} + (1-\theta)\overline{r_k^l}\right)^2\right) \leq A_1 + A_2 + A_3, \tag{17}$$

where

$$A_1 = K\theta^2 \sum_{k=1}^{K} \left(R_k^{l-1}\right)^2, \tag{18}$$

$$A_2 = K(1-\theta)^2 \sum_{k=1}^{K} \overline{r_k^l}^2, \tag{19}$$

$$A_3 = 2K\theta(1-\theta)\left(\sum_{k=1}^{K} \left(R_k^{l-1}\right)^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{K} \overline{r_k^l}^2\right)^{\frac{1}{2}}. \tag{20}$$

On the other hand, we note that

$$H_{\text{up}} = \left(\sum_{k=1}^{K} \left(\theta R_k^{l-1} + (1-\theta)\overline{r_k^l}\right)\right)^2 = B_1 + B_2 + B_3, \tag{21}$$

where

$$B_1 = \theta^2 \left(\sum_{k=1}^{K} R_k^{l-1}\right)^2, \tag{22}$$

$$B_2 = (1-\theta)^2 \left(\sum_{k=1}^{K} \overline{r_k^l}\right)^2, \tag{23}$$

$$B_3 = 2\theta(1-\theta)\left(\sum_{k=1}^{K} R_k^{l-1}\right)\left(\sum_{k=1}^{K} \overline{r_k^l}\right). \tag{24}$$

Then, according to Eq. (17) and Eq. (21), the following expression is derived

$$H = \frac{H_{\text{up}}}{H_{\text{down}}} \geq \frac{B_1 + B_2 + B_3}{A_1 + A_2 + A_3}. \tag{25}$$

Observing Eq. (25), we can conclude that $H \geq J_T$ is obtained if the following equation holds

$$\frac{B_1}{A_1}, \frac{B_2}{A_2}, \frac{B_3}{A_3} \geq J_T, \tag{26}$$

because Eq. (26) implies $B_1 \geq J_T A_1$, $B_2 \geq J_T A_2$ and $B_3 \geq J_T A_3$, which further indicate

$$H \geq \frac{B_1 + B_2 + B_3}{A_1 + A_2 + A_3} \geq \frac{J_T A_1 + J_T A_2 + J_T A_3}{A_1 + A_2 + A_3} \geq J_T. \tag{27}$$

Therefore, in the remainder of this proof, we show that Eq. (26) is satisfied. To see this, we firstly derive that $\frac{B_1}{A_1} \geq J_T$ and $\frac{B_2}{A_2} \geq J_T$ hold due to

$$\frac{\left(\sum\limits_{k=1}^{K} R_k^{l-1}\right)^2}{K \sum\limits_{k=1}^{K} \left(R_k^{l-1}\right)^2} \geq J_T \tag{28}$$

and

$$J^l = \frac{\left(\sum\limits_{k=1}^{K} \overline{r_k^l}\right)^2}{K \sum\limits_{k=1}^{K} \overline{r_k^l}^2} \geq J_T, \tag{29}$$

respectively. Further, since $\frac{B_3}{A_3} = \sqrt{\frac{B_1}{A_1} \cdot \frac{B_2}{A_2}}$, we get $\frac{B_3}{A_3} \geq J_T$. Now, the proof of this theorem is completed. ■

Theorem 1 indicates that the constraint in (8) is guaranteed by (7). As a result, Eq. (8) is a redundant constraint and we can remove it without influencing the optimal solution.

D. Problem decomposition

Now, let us reorganize the objective function of **P2** as

$$\max \sum_{k=1}^{K} R_k^L = \max \sum_{l=1}^{L} \left(\frac{1}{L} \sum_{k=1}^{K} \sum_{m=1}^{M} r_{k,m}^l p_{k,m}^l \right), \quad (30)$$

which means that the objective is to maximize the sum of all blocks' short-term throughput multiplied by $\frac{1}{L}$. On the other hand, none of the constrains in Eq. (10), Eq. (11) and Eq. (7) couples the resource allocations in different blocks. Therefore, to obtain the optimal solution of **P2**, we can sequentially performs resource allocation by maximizing the sum of short-term user throughput in each block subject to the individual involved constraints [11]. In particular, resource allocation in the lth ($l = 1, 2, \cdots, L$) block is to solve the following optimization problem

$$\mathbf{P}_l : \quad \max \sum_{k=1}^{K} \sum_{m=1}^{M} r_{k,m}^l p_{k,m}^l \quad (31)$$

$$\text{s.t.} \quad (10), (11), (7), \forall m, k. \quad (32)$$

E. Problem solution

It is evident that $\mathbf{P}_l$ is a Second Order Cone Programming (SOCP) problem in each block that can be efficiently solved[3]. In practical systems, we utilize the optimal solution $p_{k,m}^l$ to determine $Q_{k,m}^l$ in problem **P1**. Particularly, we let $Q_{k,m}^l$ equal to $Np_{k,m}^l$ rounded to the nearest integer.

According to the Law of Large Numbers, a larger N brings about smaller errors caused by the relaxation in subsection III-B. For small N, in order to illustrate the practicability of our proposed method, we present the following theorem.

Theorem 2: Of the $K \times M$ optimal variables of the optimal solution of problem $\mathbf{P}_l$, the Proportion of Integer Optimal Variable (PIOV) is larger than or equal to $1 - \frac{2(K-1)}{KM}$ if $K \leq M$ and $1 - \frac{N+K-1}{KM}$ otherwise.

The proof of Theorem 2 is omitted here owing to the limitation of space and we will present the details in a future paper. According to Theorem 2, we can see that even there are 10 users and 20 subchannels in the system, the proportion of integer optimal variables is at most 91%. Therefore, the relaxation will not introduce large deviations from the optimal system throughput of problem $\mathbf{P}_l$ as well as the threshold of Jain's index. In the next section, the numerical results are presented.

IV. Numerical Results

We consider a single cell downlink OFDMA system with $K = 20$ users and $M = 40$ subchannels. The length of an OFDMA slot is 5ms. In the following, we evaluate the PIOV of problem $\mathbf{P}_l$ and the system performance by the proposed method.

[3]For example, we can utilize CVX package developed by Grant and Boyd to solve this SOCP problem [11], [12].

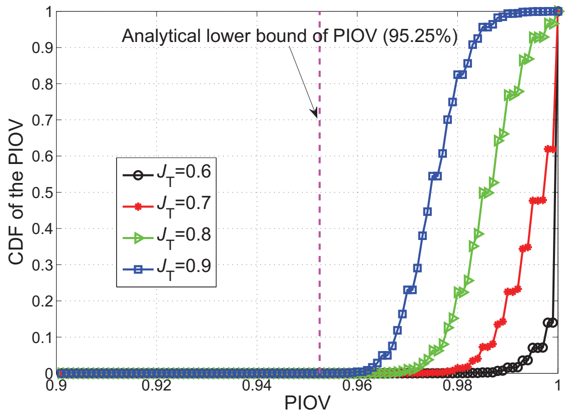

Fig. 1. The CDF curves of PIOV under several Jain's index thresholds.

TABLE I
Long-term Jain's index under several thresholds

J_T	0.6	0.7	0.8	0.9
J_{Long} of relaxed optimal	0.9919	0.9956	0.9969	0.9993
J_{Long} with $N = 10$	0.9919	0.9956	0.9969	0.9993
J_{Long} with $N = 100$	0.9919	0.9956	0.9969	0.9993

A. PIOV Evaluation

For problem $\mathbf{P}_l$, we randomly pick 10^4 blocks, where $h_{k,m}^l$ varies over blocks and subchannels with $\mathcal{CN}(0, 1)$. In Fig. 1, the Cumulative Distribution Function (CDF) curves of the PIOV under several J_T are plotted. It can be seen that the PIOV is greater than the analytical lower bound 95.25%. According to this fact, the errors caused by the relaxation is small, which will be illustrated in the next subsection.

B. Performance of the proposed method

In the simulation, long-term performance is measured by observing $L = 100$ blocks. We evaluate system performance of the optimal relaxed solution and the proposed method under $N = 10$ and $N = 100$. In Fig. 2 and Fig. 3, the short-term throughput and Jain's index in the first 50 blocks are illustrated under $J_T = 0.6$, respectively. Fig. 4 and Fig. 5 present the case $J_T = 0.9$. According to these results, it can be derived that the proposed method approaches the optimal relaxed solution when N becomes large such as $N = 100$. For small N, e.g., $N = 10$, the proposed method achieves a good suboptimal point because the deviations from the optimal relaxed throughput and the Jain's index constraint are very small. In Table I, the long-term Jain's index is presented, which confirms the conclusion in Theorem 1.

V. Conclusion

Short-term sum throughput maximization subject to short-term Jain's index constraint is performed to maximize long-term system throughput with both short-term and long-term Jain's index constraints in single cell downlink OFDMA systems. Numerical results showed that the proposed method

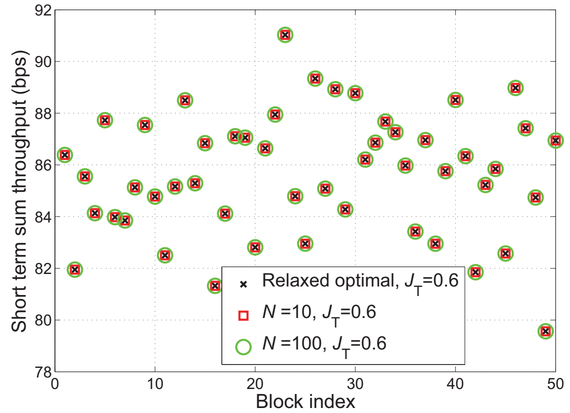

Fig. 2. Short-term throughput in the first 50 blocks with Jain's index threshold as $J_T = 0.6$.

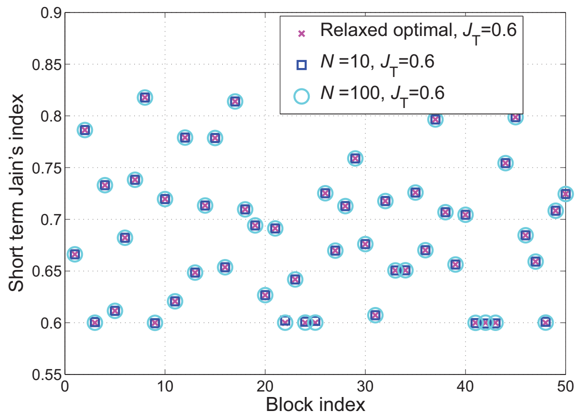

Fig. 3. Short-term Jain's index in the first 50 blocks with Jain's index threshold as $J_T = 0.6$.

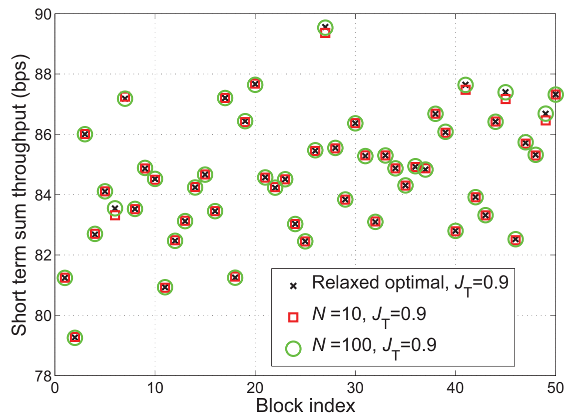

Fig. 4. Short-term throughput in the first 50 blocks with Jain's index threshold as $J_T = 0.9$.

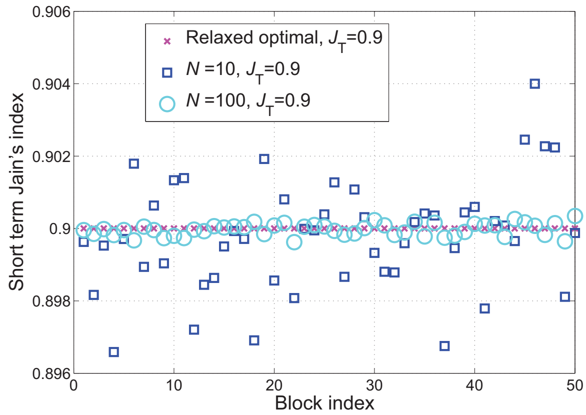

Fig. 5. Short-term Jain's index in the first 50 blocks with Jain's index threshold as $J_T = 0.9$.

achieves a good suboptimal point with small deviations from the optimal relaxed throughput and the Jain's index constraint.

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